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On Extrapolation Spaces and a.e. Convergence of Fourier Series

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A new, unified approach to recent end point estimates for the maximal operator of partial sums of Fourier series is obtained through the use of extrapolation theory. The method involves characterizing certain extrapolation spaces associated with scales of Lorentz–Zygmund spaces. © 1995 Academic Press, Inc.

1. INTRODUCTION

A natural problem in classical harmonic analysis is to describe the class of functions C with a.e. convergent Fourier series. The celebrated results of Carleson and Hunt (cf. [5], [7]) show that $L^p(T) \subset C$, $p > 1$. These results were further sharpened by Carleson and Sjölin [13] and more recently by Sjölin [14] and Soria [15, 16] to include certain classes of Orlicz spaces close to L^1. These refinements are based in extensions of the extrapolation theorem of Yano to operators of weak type and then applied to the Carleson–Hunt estimates for the maximal operator $S(f) = \sup_n |S_n(f)|$, where $S_n(f)$ denotes the partial sum of the Fourier series of f. A somewhat related extrapolation method is proposed by Taibleson and Weiss through the use of "Block spaces" (cf. [18]), which, in particular, allows to add weak type (1, 1) estimates.

In this paper we present a new and unified approach to these results through the use of the recently developed theory of extrapolation spaces (cf. [8]). In particular we show that the results of [16] can be extrapolated from the estimates of [14]. In turn the estimates of [14] can be extrapolated from the classical estimates of Carleson–Hunt. This is accomplished through the identification of certain extrapolation spaces associated with Lorentz–Zygmund spaces. Given the potential applicability of these results, we also give a general method to identify relevant extrapolation

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spaces for arbitrary scales of spaces. For other applications of extrapolation theory to problems in Fourier analysis and Approximation theory, we refer to [8, 9]. We also note that calculations of extrapolation spaces similar to those presented in Section 7 are useful in the theory of logarithmic Sobolev inequalities (cf. [11]).

The paper is organized as follows. In Section 2 we give some background on extrapolation theory and identify certain extrapolation spaces associated with the $L(\mathrm{Log}\, L)^{\theta}$ spaces, which we then use in Sections 4 and 5 to give proofs of the estimates for the maximal operator of Fourier sums described in Section 3. The last two sections, which could be skipped by a reader only interested in the estimates for the maximal operator of partial sums of Fourier series, deal with some theoretical questions that arise from our development. In Section 6 we compare the extrapolation results with some classical interpolation theorems, and in Section 7 we extend the calculations of Section 2 to arbitrary scales of spaces and relate them to generalized "reiteration theorems" for extrapolation spaces.

2. Extrapolation Theory

In this section we give a brief review of the basic background on extrapolation theory and refer to [8] for a detailed account as well as complete bibliography.

The point of departure of extrapolation theory is the extrapolation theorem of Yano [19] which can be described as follows. Suppose that T is a bounded linear operator on $L^p(0,1)$ for $p > 1$ with $\|T\|_{L^p \to L^p} = ((p-1)^{-\alpha})$, as $p \to 1$, for some $\alpha > 0$; then these estimates can be extrapolated to

$$T\colon L(\mathrm{Log}\, L)^{\alpha} \to L^1.$$

There is also a dual statement for operators T acting on $L^p(0,1)$ for p close to ∞, with $\|T\|_{L^p \to L^p} = (p^{\alpha})$, as $p \to \infty$, for some $\alpha > 0$; then T: $L^{\infty} \to \mathrm{Exp}\, L^{1/\alpha}$. The theory seeks to extend these basic L^p results to general scales of spaces. More generally, extrapolation theory aims to provide general methods to study limiting spaces and estimates in analysis.

Following [8], it is convenient to normalize the real methods of interpolation $(.,.)_{\theta,q;K}$ and $(.,.)_{\theta,q;J}$ so that they become exact interpolation functors of order θ. This is achieved by multiplying the usual norms by suitable constants. For example, the normalized $\bar{A}_{\theta,q;K}$ norm is given by $\|f\|_{\bar{A}_{\theta,q;K}} = \{(1-\theta)\theta q \int_0^{\infty}(t^{-\theta}K(t,f;\bar{A}))^q(dt/t)\}^{1/q}$.

A basic result is contained in the following corollary of [8].

THEOREM 1. (i) *Let* $\{A_\theta\}_{\theta\in(0,1)}$ *be an ordered family of Banach spaces* (*i.e. there exists* A_0 *and* A_1 *such that* $A_1 \subset A_\theta \subset A_0$, *and moreover if* $\theta_2 < \theta_1$ *then with norm one* $A_{\theta_1} \subset A_{\theta_2}$), *let* X *be a Banach space* (*or a quasi-Banach space with* $\|\cdot\|_X$ *satisfying the triangle inequality*), *and let* T *be an operator* $T\colon A_\theta \to X$, *with norm* $= O(\theta^{-\alpha})$ *as* $\theta \to 0$, $\alpha > 0$, *then*

$$T\colon \sum_{\theta\in(0,1)} \{\theta^{-\alpha}A_\theta\} \to X,$$

where

$$\sum_{\theta\in(0,1)} \{\theta^{-\alpha}A_\theta\} = \left\{a\colon a = \sum_{\theta\in(0,1)} a_\theta, \text{ (in } A_0), \ a_\theta \in A_\theta, \text{and } \sum_{\theta\in(0,1)} \theta^{-\alpha}\|a_\theta\|_{A_\theta} < \infty\right\}$$

$$\|a\|_{\Sigma_{\theta\in(0,1)}(\theta^{-\alpha}A_\theta)} = \inf\left\{\sum_{\theta\in(0,1)} \theta^{-\alpha}\|a_\theta\|_{A_\theta}\colon a = \sum_{\theta\in(0,1)} a_\theta\right\}.$$

(ii) *Suppose that the spaces* A_θ *are obtained by interpolation from a pair of mutually closed spaces* (A_0, A_1) *by the complex method or by the real method. Then, we have*

$$\sum_{\theta\in(0,1)} \{\theta^{-\alpha}A_\theta\} = \bar{A}_{(\alpha);K}$$

$$= \left\{a\colon \|a\|_{(\alpha);K} = \int_0^1 K(t, a; \bar{A})\left[\log\frac{1}{t}\right]^{\alpha-1}\frac{dt}{t} < \infty\right\}.$$

(iii) *If we are dealing with lattices*, (i) *holds for quasilinear operators.*

Let us note that if $\{A_\theta\}$ is an ordered scale (i.e., $A_{\theta_1} \subset A_{\theta_2}$, if $\theta_1 \geq \theta_2$), then to compute the spaces $\Sigma_\theta\{\theta^{-\alpha}A_\theta\}$ it is enough to consider sequences of the form $\theta_\nu = 2^{-\nu}$, $\nu \in N$. In other words we have

$$\sum_\theta \{\theta^{-\alpha}A_\theta\} = \sum_{\nu=1}^{\infty} \{2^{\nu\alpha}A_2^{-\nu\alpha}\}.$$

Indeed, let $a \in \Sigma_\theta\{\sigma^{-\alpha}A_\theta\}$, then there exists a representation of $a = \sum_{\nu=1}^\infty a_{\theta_\nu}$, with $a_{\theta_\nu} \in A_{\theta_\nu}$, and such that $\|a\|_{\Sigma_{\theta\in(0,1)}(\theta^{-\alpha}A_\theta)} \approx \sum_{\nu=1}^\infty \theta_\nu^{-\alpha}\|a_{\theta_\nu}\|_{\theta_\nu}$. For each $\mu = 2, 3, \ldots$, let $E_\mu = \{\theta_\nu \in [2^{-\mu}, 2^{-\mu+1})\}$, and $E = \{\theta_\nu / \theta_\nu \notin \cup E_\mu\}$. Let $b_{2^{-\mu}} = \Sigma_{E_\mu} a_{\theta_\nu}$, $\mu = 2, 3, \ldots$, and $b_{2^{-1}} = \Sigma_E a_{\theta_\nu}$, then $a = \sum_{\mu=1}^\infty b_{2^{-\mu}}$. The first term is easily estimated using the triangle inequality and the ordering of the scale:

$$2^{-\alpha}\|b_{2^{-1}}\|_{2^{-1}} \leq c\sum_{E}\theta_\nu^{-\alpha}\|a_{\theta_\nu}\|_{2^{-1}} \leq c\sum_{E}\theta_\nu^{-\alpha}\|a_{\theta_\nu}\|_{\theta_\nu} \leq c\sum_{\nu=1}^{\infty}\theta_\nu^{-\alpha}\|a_{\theta_\nu}\|_{\theta_\nu}.$$

Similarly,

$$\sum_{\mu=1}^{\infty} 2^{\mu\alpha}\|b_{2^{-\mu}}\|_{2^{-\mu}} \leq \sum_{\mu=1}^{\infty} 2^{\mu\alpha}\sum_{E_\mu}\|a_{\theta_\nu}\|_{\theta_\nu} = \sum_{\nu=1}^{\infty}\|a_{\theta_\nu}\|_{\theta_\nu}\sum_{\mu=-\log\theta_\nu}^{-\log\theta_\nu+1} 2^{\mu\alpha}$$
$$\leq 2^{\alpha}\sum_{\nu=1}^{\infty}\theta_\nu^{-\theta}\|a_{\theta_\nu}\|_{\theta_\nu},$$

as desired.

Let us also note that the same argument shows that, if $\{A_\theta\}_{\theta\in(0,1)}$ is an ordered scale, then $\forall\theta_0 \in (0,1)$, with norm equivalence,

$$\sum_{\theta\in(0,\theta_0)}\{\theta^{-\alpha}A_\theta\} = \sum_{\theta\in(0,1)}\{\theta^{-\alpha}A_\theta\}.$$

We consider in detail some examples important for our purposes here. For a concave function φ: $[0,1)\to\mathbf{R}_+$, $\varphi(0)=0$, let $\Lambda_\varphi(0,1)$ be the Lorentz space defined by the norm

$$\|f\|_{\Lambda_\varphi} = \int_0^1 f^*(s)\,d\varphi(s).$$

For a pair of Lorentz spaces $\Lambda_{\varphi_1}, \Lambda_{\varphi_2}$, the K functional is computed in [10]

$$K(t,f;\Lambda_{\varphi_1},\Lambda_{\varphi_2}) = \int_0^1 f^*(s)\,d\min\{\varphi_1(s),t\varphi_2(s)\}. \tag{1}$$

Using (1), we get

$$\|f\|_{(\Lambda_{\varphi_1},\Lambda_{\varphi_2})_{(\alpha);K}} = \int_0^1(\log 1/t)^{\alpha-1}\int_0^1 f^*(s)\,d\min\{\varphi_1(s),t\varphi_2(s)\}\frac{dt}{t}.$$

For example, if $\varphi_1(s)=s$, and $\varphi_2(s)=1$, $s\neq 0$, $\varphi_2(0)=0$, then $\Lambda_{\varphi_1} = L^1$, and $\Lambda_{\varphi_2} = L^\infty$. In this case (1) gives the well known formula $K(t,f;L^1,L^\infty) = tf^{**}(t)$, and we obtain (cf. [8])

$$\sum_{p>1}\{(p-1)^{-\alpha}L^p(T)\} = (L^1,L^\infty)_{(\alpha);K} = L(\operatorname{Log} L)^\alpha(T).$$

In our application to Fourier series we need to explicitly compute some extrapolation spaces associated with Lorentz spaces.

THEOREM 2. $\Sigma_\theta\{\theta^{-1}L(\text{Log } L)^{1+\theta}(T)\} = L \text{ Log } L(\text{Log Log } L)(T)$.

Proof. In our calculation it is important to represent the spaces $L(\text{Log } L)^{1+\theta}(T)$ as interpolation spaces and we need to be careful about the constants appearing in the equivalences of the norms involved. Therefore, we shall first give a detailed proof of the known fact that

$$L(\text{Log } L)^{1+\theta}(T) = \left[L \text{ Log } L(T), L(\text{Log } L)^2(T)\right]_\theta \tag{2}$$

with norm equivalence independent of θ. Note that the spaces $L(\text{Log } L)^\beta(T)$, $\beta \in [0, 2]$, have absolutely continuous norms. Then, *isometrically* (cf. [2, p. 125])

$$\begin{aligned}&\left[L \text{ Log } L(T), L(\text{Log } L)^2(T)\right]_\theta\\ &\qquad = \left[L \text{ Log } L(T)\right]^{1-\theta}\left[L(\text{Log } L)^2(T)\right]^\theta, \qquad \theta \in [0,1]. \end{aligned}\tag{3}$$

To compute the spaces appearing on the right hand side of (3) we use another technique of Calderón [2]. First observe that, since we are dealing with spaces on a finite measure space, only *large values* of the Young's functions involved are important. In our case we shall take $x \geq e^e$. Using Calderón's notation, we write $L(\text{Log } L)^\beta = A_\beta^{-1}(L^1)$, where $A_\beta(x) = x(\log x)^\beta$, $\beta \in [0, 2]$, for $x \geq e^e$. Although the precise value of A_β^{-1} is not readily available, we have that $\varphi_\beta(x) = x(\log x)^{-\beta}$ is equivalent to the inverse for large x. More precisely, an elementary computation shows that

$$\frac{x}{9} \leq \varphi_\beta\left(A_\beta(x)\right) \leq x, \qquad \text{for } x \geq e^e, \beta \in [0, 2]. \tag{4}$$

Then, according to [2, p. 166], we have, with norm equivalence independent of θ,

$$\begin{aligned}\phi_\theta^{-1}(L^1) &= \left(A_1^{-1}(L^1)\right)^{1-\theta}\left(A_2^{-1}(L^1)\right)^\theta\\ &= \left[L \text{ Log } L\right]^{1-\theta}\left[L(\text{Log } L)^2\right]^\theta, \qquad \theta \in [0,1]\end{aligned}\tag{5}$$

where

$$\phi_\theta^{-1}(x) = \left(A_1^{-1}\right)^{1-\theta}(x)\left(A_2^{-1}\right)^\theta(x) \tag{6}$$

Therefore, combining (6) and (4), we obtain, for large values of x,

$$\begin{aligned}\phi_\theta^{-1}(x) &\approx \left(x(\log x)^{-1}\right)^{1-\theta}\left(x(\log x)^{-2}\right)^{\theta}\\ &\approx x(\log x)^{-(1+\theta)}, \qquad \theta \in [0,1]. \qquad (7)\end{aligned}$$

Consequently, combining (7) with (5) and then with (4) once again, we obtain (2).

Thus, by Theorem 1 (ii)

$$\begin{aligned}\sum_\theta \left\{\theta^{-1} L(\mathrm{Log}\, L)^{1+\theta}(T)\right\} &= \left(L\,\mathrm{Log}\, L,\, L(\mathrm{Log}\, L)^2\right)_{(1);K}\\ &= \left\{ f \Big/ \int_0^1 K\left(t, f; L\,\mathrm{Log}\, L, L(\mathrm{Log}\, L)^2\right)\frac{dt}{t} < \infty \right\}\end{aligned}$$

Since the spaces $L\,\mathrm{Log}\, L$ and $L(\mathrm{Log}\, L)^2$ are Lorentz spaces the K functional can be computed using (1) and we get

$$\begin{aligned}&K\left(t, f; L\,\mathrm{Log}\, L, L(\mathrm{Log}\, L)^2\right)\\ &\approx \int_0^1 f^*(s)\, d\min\left\{\int_0^s\left(1+\log\frac{1}{u}\right)du,\ t\int_0^s\left[1+\log\frac{1}{u}\right]^2 du\right\}\end{aligned}$$

It follows that for $t < e^{-1}$,

$$\begin{aligned}&K\left(t, f; L\,\mathrm{Log}\, L, L(\mathrm{Log}\, L)^2\right)\\ &\approx \int_0^{e^{-(1-3t)/t}} f^*(s)\left(1+\log\frac{1}{s}\right)ds\\ &\quad + t\int_{e^{-(1-3t)/t}}^{e^{-1}} f^*(s)\left[1+\log\frac{1}{s}\right]^2 ds\end{aligned}$$

Integrating with respect to dt/t yields

$$\|f\|_{(L\,\mathrm{Log}\, L,\, L(\mathrm{Log}\, L)^2)_{(1);K}} \approx \int_0^{e^{-1}} f^*(s)\left(1+\log\frac{1}{s}\right)\left(\log\left(\log\frac{1}{s}\right)\right)ds,$$

from which it follows that

$$\|f\|_{(L\,\mathrm{Log}\, L,\, L(\mathrm{Log}\, L)^2)_{(1);K}} \approx \|f\|_{L\,\mathrm{Log}\, L(\mathrm{Log}(\mathrm{Log}\, L))},$$

as desired.

Remark. For a general version of Theorem 2, see the Example at the end of Section 7.

3. A Review of Estimates for the Maximal Operator

We shall present an extrapolation approach to results associated with the maximal operator on partial sums of Fourier series. Therefore, we start by presenting a brief summary of the results under consideration. The fundamental results in this area are due to Carleson [5] and Hunt [7].

Theorem 3. *Let* $S(f) = \sup_n |S_n(f)|$, *where* $S_n(f)$ *denotes the nth partial sum of the Fourier series of* f, $1 < p < \infty$, *then for every* f *of the form,* $f = g\chi_F$, *with* $2^{-1} < g \leq 1$, *we have*

$$\sup_{t>0} t^{1/p}(Sf)^*(t) \leq c_p \|f\|_p,$$

where $c_p = O(1/(p-1))$.

Using Theorem 3, the fact that $L^{p,\infty}(T) \subset L^1(T)$ with norm $O(1/(p-1))$ as $p \to 1$, with the extrapolation theorem of Yano, allows Hunt to conclude (cf. [7])

Theorem 4. S: $L(\mathrm{Log})^2(T) \to L^1(T)$.

Carleson and Sjölin (cf. [13]), C. P. Calderón [4], and Soria [15] improve on Theorem 4 by extrapolating without using the embedding of weak L^p into L^1 which worsens the constants by a factor of $(p-1)^{-1}$. The point apparently is that although Minkowski's inequality fails for $\|\cdot\|_{L^{1,\infty}}$, weak L^1 estimates can be added in the sense that (cf. [17, 3])

$$\left\| \sum_{i=1}^{\infty} c_i f_i \right\|_{L^{1,\infty}} \leq c \|\{c_i\}\|_{l\log l(N)} \sup_{i\in N} \|f_i\|_{L^{1,\infty}}.$$

These authors combine all this with the embedding of weak $L^p(T)$ into weak $L^1(T)$ to prove (cf. [15])

Theorem 5. *If* $f \in L\,\mathrm{Log}\,L(\mathrm{Log}\,\mathrm{Log}\,L)(T)$ *then* $Sf \in L^{1,\infty}(T)$.

Remark. In [8], an abstract form of Theorem 5 is given through the introduction of a modified version of the Σ extrapolation functor.

Sjölin [14] extends Theorem 4 as follows.

Theorem 6. *If* $f \in L(\mathrm{Log}\,L)^{1+\theta}(T)$ *then* $Sf \in L(\mathrm{Log}\,L)^{\theta-1}(T)$, $0 < \theta \leq 1$.

In fact, a perusal of the constants in the proof shows that

$$\int_T Sf(t)(1+\log^+ Sf(t))^{\theta-1}\,dt \le C\theta^{-1}\int_T |f(t)|(1+\log^+|f(t)|)^{1+\theta}\,dt, \tag{8}$$

where C is a constant independent of θ.

Note that the end point $\theta = 1$, corresponds to Theorem 4. Finally, the most recent result along these lines is proved in [16]

THEOREM 7. *If* $f \in L \operatorname{Log} L(\operatorname{Log}\operatorname{Log} L)(T)$ *then* $Sf \in L(\operatorname{Log} L)^{-1}(T)$.

4. EXTRAPOLATION METHODS

We now give a systematic approach to the results described in Section 3. In this section we show that Theorem 7 follows by extrapolation from (8). Then in Section 5 we show that in fact Theorem 6 can be extrapolated from Theorem 3.

We collect a number of auxiliary facts needed in the proof of Theorem 7. First we observe that

$$L(\operatorname{Log} L)^{\theta-1}(T) \subset L(\operatorname{Log} L)^{-1}(T).$$

In fact, we have

$$\int_T |f(t)|[1+\log^+|f(t)|]^{-1}\,dt \le \int_T |f(t)|[1+\log^+|f(t)|]^{\theta-1}\,dt \tag{9}$$

We also need to quantify the relationship between different ways of measuring the size of a function in the $L(\operatorname{Log} L)^{\beta}(T)$ spaces.

LEMMA 1. *For* $\theta \in (0, 1/2)$, *let* $l_\theta(f) = \|f\|_{L(\operatorname{Log} L)^{1+\theta}(T)} = \int_0^1 f^*(t)[1+\log(1/t)]^{1+\theta}\,dt$. *We have*

$$\int_0^1 |f(t)|[1+\log^+|f(t)|]^{1+\theta}\,dt \le c\Gamma_\theta(f)$$

where

$$\Gamma_\theta(f) = \begin{cases} l_\theta(f), & \text{if } l_\theta(f) \le 1, \\ \dfrac{[l_\theta(f)]^2}{\theta}, & \text{if } l_\theta(f) > 1 \end{cases}, \tag{10}$$

and c is a constant independent of θ.

Proof. We may suppose that $l_\theta = l_\theta(f) < \infty$. Using the readily verified estimate $f^*(t) \le l_\theta t^{-1}$, we obtain

$$\int_0^1 |f(t)|[1 + \log^+ |f(t)|]^{\theta+1}\,dt$$

$$= \int_{\{f^*(t) \le l_\theta t^{-1}\}} f^*(t)[1 + \log^+ f^*(t)]^{1+\theta}\,dt$$

$$\le \int_0^1 f^*(t)\left[1 + \log^+ \frac{l_\theta}{t}\right]^{1+\theta} dt. \tag{11}$$

Suppose now that $l_\theta \le 1$. Estimating the function $[1 + \log^+ (l_\theta)/t)]^{1+\theta}$ separately in $(0, l_\theta)$ and $(l_\theta, 1)$ we see that the right-hand side of (11) is bounded by $4\, l_\theta$. On the other hand, if $l_\theta > 1$,

$$\int_0^1 f^*(t)\left[1 + \log^+ \frac{l_\theta}{t}\right]^{1+\theta} dt \le (1 + \log l_\theta)^{1+\theta} l_\theta$$

Now, since $\log l_\theta \le \theta^{-1} l_\theta^\theta$, and $l_\theta > 1$, $\theta \in (0, 1/2)$, we have

$$(1 + \log l_\theta)^{1+\theta} l_\theta \le \left(1 + \frac{l_\theta^\theta}{\theta}\right)^{1+\theta} \le \left(2 l_\theta^\theta\right)^{1+\theta} l_\theta \theta^{-1-\theta} \le c l_\theta^2 \theta^{-1} \theta^{-\theta}.$$

The desired result follows since $\theta \in (0, 1/2)$, and $\theta^{-\theta}$ is bounded as $\theta \to 0$.

We are now ready for the

Proof of Theorem 7. From (8), (9), and Lemma 1, we have,

$$S\colon L(\operatorname{Log} L)^{1+\theta}(T) \to L(\operatorname{Log} L)^{-1}(T),$$

with

$$\int_T |Sf(t)|[1 + \log^+ |Sf(t)|]^{-1}\,dt \le c\theta^{-1} \Gamma_\theta(f) \tag{12}$$

as $\theta \to 0$, and where $\Gamma_\theta(f)$ is defined in (10).

Observe that the functional

$$f \to l(f) = \int_T |f(t)|[1 + \log^+ |f(t)|]^{-1}\,dt \tag{13}$$

is subadditive on positive functions, and that S is a positive subadditive operator. If a norm estimate were available instead of (8) we could extrapolate directly. However, in the situation at hand we need an extra argument to overcome the nonlinearity of the estimate.

Let $f \in \Sigma_{\theta \in (0,1)}\{\theta^{-1}L(\operatorname{Log} L)^{1+\theta}(T)\} = \Sigma_{\theta \in (0,1/2)}\{\theta^{-1}L(\operatorname{Log} L)^{1+\theta}(T)\}$. Consider a nearly optimal decomposition $f = \Sigma_{\theta<1/2} f_\theta$ such that

$$\|f\|_{\Sigma_\theta\{\theta^{-1}L(\operatorname{Log} L)^{1+\theta}(T)\}} \approx \sum_{\theta<1/2} \theta^{-1}\|f_\theta\|_{L(\operatorname{Log} L)^{1+\theta}(T)}.$$

Associated with this decomposition we define the sets $E = \{\theta \in (0,1/2)\colon \|f_\theta\|_{L(\operatorname{Log} L)^{1+\theta}(T)} \le 1\}$, $F = \{\theta \in (0,1/2)\colon \|f_\theta\|_{L(\operatorname{Log} L)^{1+\theta}(T)} > 1\}$, and let

$$f = f_0 + f_1, \qquad \text{with } f_0 = \sum_{\theta\in E} f_\theta,\ f_1 = \sum_{\theta\in F} f_\theta.$$

Applying the functional l defined in (13) to Sf, taking into account that $Sf \le Sf_0 + Sf_1$, gives

$$l(Sf) \le l(Sf_0) + l(Sf_1).$$

We now estimate each of these terms separately. By subadditivity,

$$\begin{aligned} l(Sf_0) &\le \sum_{\theta\in E} l(Sf_\theta) \\ &\le c\sum_{\theta\in E} \theta^{-1}\Gamma_\theta(f_\theta) \qquad (\text{by } (12)) \\ &\le c\sum_{\theta\in E} \theta^{-1} l_\theta(f_\theta) \qquad (\text{by the definitions of } E \text{ and } \Gamma) \\ &\le c\sum_{\theta\in(0,1/2)} \theta^{-1}\|f_\theta\|_{L(\operatorname{Log} L)^{\theta+1}(T)} \approx \|f\|_{\Sigma_\theta\{\theta^{-1}L(\operatorname{Log} L)^{1+\theta}(T)\}}. \end{aligned}$$

Similarly, since the functional $f \to [l(f)]^{1/2}$ is also subadditive on positive functions, we obtain

$$\begin{aligned} [l(Sf_1)]^{1/2} &\le c\sum_{\theta\in F} [l(Sf_\theta)]^{1/2} \\ &\le c\sum_{\theta\in F} \theta^{-1/2}[\Gamma(f_\theta)]^{1/2} \qquad (\text{by } (12)) \\ &\le c\sum_{\theta\in F} \theta^{-1/2}\theta^{-1/2}\|f_\theta\|_{L(\operatorname{Log} L)^{1+\theta}(T)} \qquad (\text{by } (10)) \\ &\le c\sum_{\theta\in(0,1/2)} \theta^{-1}\|f_\theta\|_{L(\operatorname{Log} L)^{1+\theta}(T)} \approx \|f\|_{\Sigma_\theta\{\theta^{-1}L(\operatorname{Log} L)^{1+\theta}(T)\}}. \end{aligned}$$

Consequently,

$$I(Sf) \leq c\left\{\|f\|_{\Sigma_\theta\{\theta^{-1}L(\mathrm{Log}\, L)^{1+\theta}(T)\}} + \left(\|f\|_{\Sigma_\theta\{\theta^{-1}L(\mathrm{Log}\, L)^{1+\theta}(T)\}}\right)^2\right\},$$

as we wished to show.

5. Division of Inequalities

In this section we wish to point out the connection between what in [8] is described as the principle of "division of inequalities" and the estimates of Theorem 6 and Theorem 3. We shall be brief and refer the reader to [8] for more details.

Let us recall the relevant results from [8, cf. Cor. 3.14]. Let $\bar{A}$ and $\bar{B}$ be mutually closed Banach pairs, let $q(\theta)$ be an extended real valued function with $1 \leq q(\theta) \leq \infty$, and let $M(\theta)$, $N(\theta)$, be a tempered functions (i.e., $M(\theta) \approx M(2\theta)$ as $\theta \to 0$). Then, for an operator T, the following are equivalent:

(i) $T\colon \bar{A}_{\theta,1;J} \to \bar{B}_{\theta,\infty;K}$, with norm $cM(\theta)$, $\forall\theta \in (0,1)$;

(ii) $T\colon \Sigma_\theta\{t^\theta N(\theta)M(\theta)\bar{A}_{\theta,q(\theta);K}\} \to \Sigma_\theta\{t^\theta N(\theta)\bar{B}_{\theta,q(\theta);K}\}$.

The point is that we can divide our estimates by a suitable function $N(\theta)$ before applying the Σ functor and we can choose the function $N(\theta)$ as we wish. Applying this result to Theorem 3, with $L(p,\infty) = (L^1, L^\infty)_{1/p',\infty;K}$, $M(\theta) = \theta^{-2}$, $N(\theta) = \theta^{-\beta-1}$, $\beta \in (0,1)$, and taking into account that

$$\|f\|_{(L^1,L^\infty)_{1/p',\infty;K}} = \sup_{t>0} t^{1/p} f^{**}(t) \leq \frac{1}{p'} \sup_{t>0} t^{1/p} f^*(t),$$

gives (cf. [8, Example 3.15, p. 32, second formula from the top])

$$\int_0^1 \left(\log\frac{1}{s}\right)^\beta s(Sf)^{**}(s)\frac{ds}{s} \leq c\int_0^1 \left(\log\frac{1}{s}\right)^{\beta+2} sf^{**}(s)\frac{ds}{s}.$$

Integrating by parts, we recover Theorem 6. Thus Theorem 6 follows from Theorem 3.

For completeness sake we remark that with small changes the arguments given here to deal with weak type estimates decaying like θ^{-1} could be used to deal with other types of decay, e.g., like $\theta^{-\alpha}$ (cf. [15]).

6. Extrapolation vs Interpolation

It is instructive to compare the results obtained with classical interpolation theorems. As we have seen, for suitable scales, extrapolation spaces can be characterized as limiting interpolation spaces. We now consider briefly the (*interpolation like*) application of the interpolation functors that appear in extrapolation theory. In what follows we work with function spaces based on a finite measure space. In [20], Zygmund shows that if T is a quasilinear operator of weak types $(1, 1)$, and $(2, 2)$ then T: $L(\mathrm{Log}\, L)^{\theta} \to L(\mathrm{Log}\, L)^{\theta - 1}$, $0 < \theta < 1$ (cf. [6] for a proof of this result using the $(.,.)_{(1);K}$ method). Interpolation theorems of a similar nature were obtained by O'Neil [12]. For example, in [12] it is shown that

Theorem 8. *Let T be a sublinear operator of weak types $(1, p), (q, r)$, with $0 < p < r < \infty$, $1 \leq q < \infty$. Then T: $L(\mathrm{Log}\, L)^{\theta} \to L^{p, 1/\theta}$, $0 < \theta < 1$.*

We illustrate our point giving here a proof of Theorem 8 using the $(.;.)_{(1);K}$ method. In particular, our approach, which should be compared with [20, 12, and 1], can be used transform this result, and others like it, into general real interpolation theorems.

Proof of Theorem 8. Applying the $(.;.)_{(1);K}$ method, we have

$$T: (L^1, L^q)_{(1);K} \to (L^{p,\infty}, L^{r,\infty})_{(1);K}.$$

However,

$$(L^1, L^q)_{(1);K} = (L^1, L^{\infty})_{(1);K} = L\, \mathrm{Log}\, L$$

and (cf. [6] for a similar calculation with $p = 1$)

$$(L^{p,\infty}, L^{r,\infty})_{(1);K} = (L^{p,\infty}, L^{\infty})_{(1);K} \subset L^{p,1}.$$

Thus, T: $L\, \mathrm{Log}\, L \to L^{p,1}$. Interpolating this result once again with T: $L^1 \to L^{p,\infty}$, we get

$$T: (L^1, L(\mathrm{Log}\, L))_{\theta,1;K} \to (L^{p,\infty}, L^{p,1})_{\theta,1}$$

The result follows from the known characterizations

$$(L^1, L(\mathrm{Log}\, L))_{\theta,1;K} = L(\mathrm{Log}\, L)^{\theta}, (L^{p,\infty}, L^{p,1})_{\theta,1;K} \subset L^{p,1/\theta}.$$

Remark. For the maximal operator of partial Fourier sums weak type $(1, 1)$ is, of course, not available, and extrapolation was used to single out the right domain space required to arrive to the desired target space. We also point out that the interpolation theorems for Lorentz–Zygmund spaces presented in [1] can be extended in this fashion.

7. Generalized Reiteration Theorems

We now extend the results obtained in Section 2. In fact, we show that the concrete calculations of Section 2, for Lorentz–Zygmund spaces, are special cases of rather general results for interpolation (extrapolation) scales of spaces. The unifying theme is that of "reiteration."

Let us recall some results from [6]. Let $\bar{A}$ be an ordered pair of Banach spaces, i.e., $A_0 \supset A_1$, then it is shown in [6] that

$$K(t, f; A_0, (A_0, A_1)_{(1); K}) \approx t \int_{e^{-1/t}}^{1} K(u, f; \bar{A})\, du/u \tag{14}$$

and

$$K(t, f; (A_0, A_1)_{(1); K}, A_1) \approx \int_0^1 K(\min\{\varphi^{-1}(t), u\}, f; \bar{A})\, du/u, \tag{15}$$

where $\varphi(t) = t \log(e/t)$.

Integrating 15 in $(0, 1)$ with respect to dt/t and changing the order of integration yields

THEOREM 9. $((A_0, A_1)_{(1); K}, A_1)_{(1); K} = (A_0, A_1)_{(2); K}$.

We are now ready to prove the following extension of Theorem 2.

THEOREM 10.

$$((A_0, A_1)_{(1); K}, (A_0, A_1)_{(2); K})_{(1); K}$$
$$= \left\{ f \in (A_0, A_1)_{(1); K} \Big/ \int_0^1 K(s, f; \bar{A}) \left[\log^+\left(\log \frac{1}{s}\right)\right] \frac{ds}{s} < \infty \right\}.$$

Proof. The proof is by "ping-pong" iteration. In fact, successively applying Theorem 9, (14), and (15) (applied to $B_0 = (A_0, A_1)_{(1); K}$; $B_1 = (B_0, A_1)_{(1); K}$), we get

$$\begin{aligned} K(t, f; (A_0, A_1)_{(1); K}, (A_0, A_1)_{(2); K}) &\approx K(t, f; (A_0, A_1)_{(1); K}, ((A_0, A_1)_{(1); K}, A_1)_{(1); K}) \\ &\approx t \int_{e^{-1/t}}^{1} K(u, f; (A_0, A_1)_{(1); K}, A_1) \frac{du}{u} \\ &\approx t \int_{e^{-1/t}}^{1} \int_0^1 K(\min\{\varphi^{-1}(u), s\}, f; \bar{A}) \frac{ds}{s} \frac{du}{u}. \end{aligned}$$

Therefore,

$$\|f\|_{((A_0, A_1)_{(1); K}(A_0, A_1)_{(2); K})_{(1); K}} \approx \int_0^1 \int_{e^{-1/t}}^{1} \int_0^1 K(\min\{\varphi^{-1}(u), s\}, f; \bar{A}) \frac{ds}{s} \frac{du}{u}\, dt,$$

which in turn is equivalent to the sum of two integrals,

$$\int_0^1 \int_{e^{-1/t}}^1 \int_0^{\varphi^{-1}(u)} K(s,f;\bar{A})\frac{ds}{s}\frac{du}{u}\,dt$$
$$+\int_0^1 \int_{e^{-1/t}}^1 \int_{\varphi^{-1}(u)}^1 K(\varphi^{-1}(u),f;\bar{A})\frac{ds}{s}\frac{du}{u}\,dt$$
$$= (I) + (II), \quad \text{say}.$$

Now

$$\begin{aligned}(II) &= \int_0^1 \int_{e^{-1/t}}^1 K(\varphi^{-1}(u),f;\bar{A})\log\frac{1}{\varphi^{-1}(u)}\frac{du}{u}\,dt \\ &\le \int_0^1 \left[\frac{K(\varphi^{-1}(u),f;\bar{A})}{\varphi^{-1}(u)}\right]\left[\varphi^{-1}(u)\log\frac{e}{\varphi^{-1}(u)}\right]\left(\log\frac{1}{u}\right)^{-1}\frac{du}{u} \\ &= \int_0^1 \left[\frac{K(\varphi^{-1}(u),f;\bar{A})}{\varphi^{-1}(u)}\right](\log 1/u)^{-1}\,du \\ &= \int_0^1 \left[\frac{K(u,f;\bar{A})}{u}\right]\left(\log\frac{1}{\varphi(u)}\right)^{-1}\varphi'(u)\,du \\ &\le c\|f\|_{(A_0,A_1)_{(I);K}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(I) &= \int_0^1 K(s,f;\bar{A})\int_{\varphi(s)}^1 \int_0^{1/(\log(1/u))} dt\frac{du}{u}\frac{ds}{s} \\ &= \int_0^1 K(s,f;\bar{A})\int_{\varphi(s)}^1 \frac{1}{\log(1/u)}\frac{du}{u}\frac{ds}{s} \\ &\le \int_0^1 K(s,f;\bar{A})\log^+\left(\log\frac{1}{\varphi(s)}\right)ds \\ &\approx \int_0^1 K(s,f,\bar{A})\log^+\left(\log^+\left(\log\frac{1}{s}\right)\right)ds \\ &\quad + \int_0^1 K(s,f,\bar{A})\log^+\left(\log^+\frac{1}{s}\right)ds,\end{aligned}$$

and the desired result follows.

One should view Theorem 10 as an abstract version of Theorem 2. This is developed in detail in the following example.

EXAMPLE. Consider the pair $\bar{A} = (L^1(T), L^\infty(T))$. Then, $\bar{A}_{(1);K} = L \operatorname{Log} L(T)$, $\bar{A}_{(2);K} = L(\operatorname{Log} L)^2(T)$, and by Theorem 10, $(\bar{A}_{(1);K}, \bar{A}_{(2);K})_{(1);K} = L \operatorname{Log} L \operatorname{Log} \operatorname{Log} L(T)$. Combining this calculation with $L(\operatorname{Log} L(T))^{1+\theta} = [\bar{A}_{(1);K}, \bar{A}_{(2);K}]_\theta$ (see (2)) and Theorem 1 (ii), we obtain Theorem 2.

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